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$1/N^2$ expansion of the mean field for lattice chiral and gauge models

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Abstract. For lattice chiral and gauge models we develop an $1/N^2$ expansion of the mean-field approximation. Special attention is paid to the free energy for which we also present as an $1/N^2$ expansion the effect of fluctuations around the mean-field solution. The differences between U(N) and SU(N) are pointed out. Finally, for the chiral model we apply the mean-field saddle-point technique to compute the two-point correlation function.

1. Introduction

The lattice description of models in physics is now very well developed. When applied to field theory, the lattice approach becomes particularly relevant in the neighbourhood of a critical point; for many theories under consideration by particle physicists the interesting critical point corresponds to $1/\beta \rightarrow 0$ where β is the strong coupling constant (inverse temperature). Therefore, a good understanding of the weak-coupling behaviour of the theory is clearly necessary.

Apart from the standard $1/\beta$ expansion, other techniques have been developed which are also able to describe the weak-coupling phase of lattice models. In the absence of exact results they consist of expansions in other parameters, namely in $1/N$ ('t Hooft 1974, Witten 1979) or in the case of the mean-field approximation (MF) (Brezin and Drouffe 1982) in $1/d$ where d is the spacetime dimension. In general these non-perturbative approaches provide good control of the behaviour of the theory in the large-β region and even in the intermediate region which is crucial for the discovery of an eventual phase transition.

The implications of the MF for lattice chiral models were first studied by Kogut *et al* (1982); more recently an improved MF was obtained by several groups (Brihaye and Rossi 1984a, Guha and Lee 1984). Most of the information is available for a few finite values of $N (N=2,3,4,5)$ or in the large-N limit. The MF method was also applied to lattice gauge theories. A naive and improved MF approximation was obtained by considering the mass matrix determinant of the fluctuations around the MF (Müller and Rühl 1982, Hasegawa and Yang 1982). A detailed analysis of the free energy was performed in the critical region of the coupling constant, i.e. the region where the MF prediction (weak coupling) intersects the strong-coupling curves (Müller *et al* 1983). This analysis provides important information on the phase structure of the model.

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More recently, finite-temperature effects and quarks were included in the computation of the free energy (Green and Karsch 1984).

Another aspect of the MF is to use it to get information about the standard $1/\beta$ perturbative expansion without calculating any Feynman graph. We know that this expansion is plagued with a number of momentum integrals which cannot always be computed analytically. So far, the naive MF can lead to the large-d behaviour of these integrals (d being the dimension of spacetime) while an improved MF gives the $1/d$ corrections.

The purpose of this paper is to compute the $1/\beta$ and $1/\beta^2$ coefficients of the free energy for the lattice chiral model and for the Wilson action in an axial gauge by the MF method (naive and improved). The large N, $1/N^2$ and $1/N^4$ corrections for the $U(N)$ and $SU(N)$ symmetry groups are computed using the so-called 'single-link group integral' (Brower and Nauenberg 1980, Brower *et al* 1981) under the form of an $1/N^2$ expansion (Brihaye and Rossi 1984b). This new form is used to simplify a technique developed earlier by Müller and Rühl (1982). We work with an arbitrary number of spacetime dimensions, all quantities appearing in terms of the variable βd. We check, to the best of our knowledge, the agreement of our formulae with the standard $1/\beta$ expansion and point out some restrictions in the domain of validity of the $1/N^2$ corrections.

The paper is organised as follows: in § 2, we give the single-link group integral formulae of interest for our computation; in § 3 we treat the chiral model, giving an expression for the free energy and we also show how the MF can provide an approximation for the two-point correlation functions. Finally in § 4, we use the Wilson action in the axial gauge, compute the $1/N^2$ corrections to the MF and analyse their consequence for the free energy of this model.

2. Single-link group integral

The basic ingredient in the MF approach of a lattice unitary model described by an action $S(U)$ is to substitute the constrained variable U (belonging to $U(N)$) or $SU(N)$) by an unconstrained one, say V, belonging to $GL(N, C)$. Symbolically, one has

$$Z(\beta) = \int dU \exp S(U) \tag{2.1}$$

$$Z(\beta) = \int dV\, dA \exp S_{\text{eff}}(V, A) \tag{2.2}$$

$$S_{\text{eff}}(V, A) = S(V) - N \operatorname{Tr}(AV + V^+ A^+) + N^2 w(A, A^+). \tag{2.3}$$

The matrix A is a Lagrange multiplier insuring the contraint and w is the so-called one-link group integral defined by

$$w(A, A^+) = \frac{1}{N^2} \ln \int dU \exp N \operatorname{Tr}(UA^+ + AU^+). \tag{2.4}$$

It depends on the group only, not on the model.

The mean-field prescription consists in looking for a saddle-point configuration of the effective action which is translational invariant and proportional to the identity matrix

$$\bar{A} = a\mathbb{1}, \qquad \bar{V} = v\mathbb{1}. \tag{2.5}$$

After solving the equations of motion, one studies the action (2.3) for the fluctuations around the saddle point

$$A = a\mathbb{1} + \gamma, \qquad \gamma = B + \mathrm{i}C, \qquad B = B^+, \qquad C = C^+ \tag{2.6}$$

$$V = v\mathbb{1} + \xi, \qquad \xi = D + \mathrm{i}E, \qquad D = D^+, \qquad E = E^+ \tag{2.7}$$

that is

$$S_{\mathrm{eff}} = S_{\mathrm{eff}}(a, v) + S^{(2)}(\gamma, \xi) + S^{(3)}(\gamma, \xi) + \dots. \tag{2.8}$$

In (2.6) and (2.7), we have split the fluctuations in Hermitian and anti-Hermitian parts for later convenience.

While the expansion (2.8) is straightforward for the first two terms of the effective action (2.3), the treatment of the w part requires a lot of work. Let us first present an $1/N^2$ expansion of w valid *for* $t < 1$ only (see (2.13)):

$$w(A, A^+) = w_\infty(\lambda_i) + \frac{1}{N^2} w_1(\lambda_i, \phi) + \frac{1}{N^4} w_2(\lambda_i, \phi) + \mathrm{O}\left(\frac{1}{N^6}\right) \tag{2.9}$$

$$w_\infty(\lambda_i) = \frac{2}{N} \sum_{i=1}^{N} \lambda_i^{1/2} - \frac{1}{2N^2} \sum_{i,j=1}^{N} \ln(\lambda_i^{1/2} + \lambda_j^{1/2}) - \tfrac{3}{4} \tag{2.10}$$

$$w_1(\lambda_i, \phi) = -\tfrac{1}{8} \ln(1-t) + \delta \ln \sum_{l=-\infty}^{+\infty} \exp \mathrm{i}Nl\phi(1-t)^{l^2/2} \tag{2.11}$$

$$w_2(\lambda_i, \phi) = N^2 t^{(3)} \left[\frac{3}{2^7} \frac{(1-\delta)}{(1-t)^3} + \frac{\delta}{24} \left(\sum_{l=-\infty}^{+\infty} \exp \mathrm{i}Nl\phi(l^2 - \tfrac{1}{4})(l^2 - \tfrac{9}{4})(1-t)^{l^2/2-3} \right) \right.$$
$$\left. \times \left(\sum_{l=-\infty}^{+\infty} \exp \mathrm{i}Nl\phi(1-t)^{l^2/2} \right)^{-1} \right] \tag{2.12}$$

where the following definitions have been introduced

$$t^{(i)} = \sum_{j=1}^{N} \left(\frac{1}{2N(\lambda_j)^{1/2}} \right)^i, \qquad t \equiv t^{(1)} \tag{2.13}$$

$$\phi = \frac{1}{2\mathrm{i}N} \ln \frac{\det A}{\det A^+}, \qquad \delta = \begin{cases} 1 & \text{in SU}(N) \\ 0 & \text{in U}(N) \end{cases} \tag{2.14}$$

and λ_i are the eigenvalues of the matrix AA^+. We further know the coefficient w_3 in the U(N) case. We present it here for information

$$w_3(\lambda_i) = \frac{3^2 N^4}{2^{10}} \left(5 \frac{t^{(5)}}{(1-t)^5} + 7 \frac{(t^{(3)})^2}{(1-t)^6} \right). \tag{2.15}$$

We give for further reference the quadratic and cubic fluctuations of the function w around the saddle point (2.5) (i.e. $\lambda_i = a^2$ and $\phi = 0$). They respectively read

$$w(A, A^+) = w(a\mathbb{1}, a\mathbb{1}) + \mathcal{M} + \mathcal{T}_3 + \dots \tag{2.16}$$

$$\mathcal{M} = (f_1 + f_2) \frac{\mathrm{Tr}\, B^2}{N} + f_3 \left(\frac{\mathrm{Tr}\, B}{N} \right)^2 + f_1 \frac{\mathrm{Tr}\, C^2}{N} + g_1 \left(\frac{\mathrm{Tr}\, C}{N} \right)^2 \tag{2.17}$$

$$\mathcal{T}_3=\left(\frac{1}{a}f_2+\frac{1}{3}f_5\right)\frac{\operatorname{Tr} B^3}{N}+\tfrac{1}{3}f_7\left(\frac{\operatorname{Tr} B}{N}\right)^3+\left(f_6+\frac{1}{a}f_3\right)\frac{\operatorname{Tr} B \operatorname{Tr} B^2}{N^2}$$
$$+\frac{1}{a}f_3\frac{\operatorname{Tr} B \operatorname{Tr} C^2}{N^2}+\frac{1}{a}f_2\frac{\operatorname{Tr} BC^2}{N}+g_2\frac{\operatorname{Tr} B}{N}\left(\frac{\operatorname{Tr} C}{N}\right)^2$$
$$-\frac{2}{a}g_1\frac{\operatorname{Tr} C}{N}\frac{\operatorname{Tr} BC}{N}. \tag{2.18}$$

As expected (2.17) and (2.18) are even functions of C; here, for shortness, we will present only the functions appearing in the quadratic part $\mathcal{M}$ and write all other ones in the appendix.

$$\frac{f_1(a)}{N}=\frac{\partial}{\partial\lambda_k}w\bigg|_{\rm sp} \tag{2.19}$$

$$\frac{f_2(a)}{N}\delta_{kl}+\frac{f_3(a)}{N^2}=2a^2\frac{\partial^2}{\partial\lambda_k\partial\lambda_l}w\bigg|_{\rm sp} \tag{2.20}$$

$$\frac{f_4(a)}{N}=\frac{2a^2}{\lambda_k-\lambda_l}\left(\frac{\partial}{\partial\lambda_k}-\frac{\partial}{\partial\lambda_l}\right)w\bigg|_{\rm sp}, \qquad k\neq l \tag{2.21}$$

$$g_1(a)=\frac{1}{2a^2}\frac{\partial^2}{\partial\phi^2}w\bigg|_{\rm sp}. \tag{2.22}$$

Using the formulae (2.9)-(2.14), one can compute the above quantities; up to $1/N^4$ terms and for $a>\frac{1}{2}$, we have obtained

$$f_1(a)=\frac{1}{a}\left(1-\frac{1}{4a}\right)-\frac{1}{2^4N^2a^2(2a-1)}+\frac{\delta}{2aN^2}\dot{s}(a) \tag{2.23}$$

$$f_2(a)=f_4(a)=\frac{1}{a}\left(\frac{3}{8a}-1\right)+\frac{3}{2^4N^2a^2(2a-1)}-\frac{3\delta a}{2N^2}\dot{s}(a) \tag{2.24}$$

$$f_3(a)=\frac{1}{8a^2}+\frac{1}{16N^2a^2(2a-1)^2}+\frac{\delta}{2N^2}\left(\ddot{s}(a)+\frac{2}{a}\dot{s}(a)\right) \tag{2.25}$$

$$g_1(a)=-\frac{\delta(2a-1)}{aN^2}\dot{s}(a) \tag{2.26}$$

and we have introduced the function $s(a)$ related to the θ_3 theta function:

$$s(a)=\ln\sum_{l=-\infty}^{+\infty}\left(1-\frac{1}{2a}\right)^{l^2/2}$$
$$=\ln\theta_3(0,(1-1/2a)^{1/2}). \tag{2.27}$$

The function $s(a)$ behaves straightforwardly around $a=\frac{1}{2}$ and its asymptotic behaviour (for $a\to\infty$) can be identified after use of the following property of theta function (Doetsch 1955):

$$\sum_{n=-\infty}^{+\infty}\exp -n^2\pi z|_{z\to 0}=\sqrt{\frac{\pi}{z}}+O(|z|^{\gamma}) \tag{2.28}$$

for γ arbitrarily large. Combining (2.27) and (2.28) leads to the following approximation

$$s(a)|_{a\to\infty} = -\tfrac{1}{2}\ln[-(1/2\pi^2)\ln(1-1/2a)] + \ln(1+\mathrm{O}(|z|^\gamma)). \tag{2.29}$$

This approximation was used numerically in Guha and Lee (1984); in the following sections it will be confirmed that it contains all the perturbation information for the models under investigation.

It must be noticed that the term $\mathcal{M}$ (see (2.17)) was already obtained in Guha and Lee (1982), Müller and Rühl (1982) and Hasegawa and Yang (1983) using an indirect method. Here, we present it again for completeness and to show how its derivation works in our formalism. Moreover, our formulae are more suitable for obtaining higher-order corrections in N^2.

3. Chiral model

The action for the lattice chiral model is taken to be

$$S(U) = N\beta \sum_{\mu,n} \mathrm{Tr}(U_n U^+_{n+\mu} + U^+_n U_{n+\mu}) \tag{3.1}$$

and the associated effective action follows from (2.3)

$$S_{\mathrm{eff}}(V, A) = \sum_n \beta N \sum_\mu \mathrm{Tr}(V_n V^+_{n+\mu} + V^+_n V_{n+\mu}) - N\,\mathrm{Tr}(A_n V^+_n + V_n A^+_n) + N^2 w(A_n, A^+_n). \tag{3.2}$$

Using the ansatz (2.5) we obtain the following set of saddle-point equations

$$a = 2\beta d v \qquad v = \tfrac{1}{2}(\partial/\partial a) w \qquad (\partial/\partial\phi) w = 0, \tag{3.3}$$

the third equation being satisfied by $\phi = 0$ (or, equivalently, a is real). For $a > \frac{1}{2}$, one can use the expansion (2.9) for w and arrange the system (3.3) in the following form

$$a = 2\beta d v \tag{3.4a}$$

$$\frac{a^2}{2\beta d} - a + \frac{1}{4} = \frac{1}{N^2}\left(\frac{-1}{16(2a-1)} + \frac{\delta a}{2}\dot{s}(a)\right) + \frac{1}{N^4}\left(\frac{-9a}{2^7(2a-1)^4} + \ldots\right) + \mathrm{O}\left(\frac{1}{N^6}\right) \tag{3.4b}$$

assuming that the function a admits an $1/N^2$ expansion; we have found the formal solution below

$$a = a_0 + a_1/N^2 + a_2/N^4 + \mathrm{O}(1/N^4) \tag{3.5}$$

$$a_0 = \beta d(1 \pm (1 - 1/2\beta d)^{1/2}) \tag{3.6a}$$

$$a_1 = a_0^2 \dot{w}_1(a_0)/(2a_0 - 1) \tag{3.6b}$$

$$a_2 = [a_0^2/(2a_0-1)](\dot{w}_2(a_0) + a_1 \ddot{w}_1(a_0) - a_1^2/2a_0^2) \tag{3.6c}$$

and the corresponding expansion for the free energy takes the form

$$\frac{F_{\mathrm{eff}}}{VN^2} \equiv \ln Z = a_0 - \tfrac{1}{2} - \tfrac{1}{2}\ln 2a_0 + \frac{1}{N^2} w_1(a_0) + \frac{1}{N^4}\left(\frac{a_0^2 \dot{w}_1^2(a_0)}{2(2a_0-1)} + w_2(a_0)\right) + \mathrm{O}\left(\frac{1}{N^6}\right). \tag{3.7}$$

Let us insist on the fact that all coefficients in (3.5) and (3.7) are N independent. In $\mathrm{U}(N)$ however the expansions above simplify drastically; in particular, (3.7) can be written

$$\frac{F_{\rm eff}}{VN^2}=a_0-\tfrac{1}{2}-\tfrac{1}{2}\ln 2a_0-\frac{1}{8N^2}\ln\left(1-\frac{1}{2a_0}\right)+\frac{1}{2^5N^4(2a_0-1)^3}+O\left(\frac{1}{N^6}\right). \tag{3.8}$$

A few remarks should be made before continuing: first the solutions (3.5)-(3.6) are defined only in the region $\beta d \geq \frac{1}{2}$ (weak coupling); the solution corresponding to the minus sign in (3.6) has to be rejected because it does not obey the condition $a > \frac{1}{2}$.

While the large N limit of a (i.e. a_0) and of $F_{\rm eff}$ are defined on the interval $\beta d \geq \frac{1}{2}$, the $1/N^2$ corrections become infinite when βd approaches the critical point ($\beta d = \frac{1}{2}$, $a_0 = \frac{1}{2}$). The singularities are stronger while increasing the order in $1/N^2$. Therefore, the $1/N^2$ corrections have to be considered seriously on a smaller interval; this peculiarity reflects the fact that the expansion (2.9) is unable to reproduce the w function around the phase transition point arising in the large-N limit. This problem has already been raised in Müller and Rühl (1982) and Goldschmidt (1980).

In the large βd region, things are smooth and analytic; we obtained the following expansions respectively in U(N) and SU(N)

$$\frac{F_{\rm eff}}{VN^2}=2\beta d-\tfrac{1}{2}\ln 2\beta d+\frac{N^2+1}{2^5N^2}\frac{1}{\beta d}+\frac{N^2+2}{2^8N^2}\left(\frac{1}{\beta d}\right)^2$$
$$+\frac{10N^4+25N^2+6}{3\cdot 2^{12}\cdot N^4}\left(\frac{1}{\beta d}\right)^3+O\left(\frac{1}{\beta d}\right)^4+O\left(\frac{1}{N^6}\right) \tag{3.9}$$

$$\frac{F_{\rm eff}}{VN^2}=2\beta d-\frac{N^2-1}{N^2}\ln 2\beta d+\frac{N^2-3}{2^5N^2}\frac{1}{\beta d}+\frac{N^2-\frac{14}{3}}{2^8N^2}\left(\frac{1}{\beta d}\right)^2+O\left(\frac{1}{\beta d}\right)^3+O\left(\frac{1}{N^4}\right) \tag{3.10}$$

whose $1/\beta d$ coefficients coincide with standard $1/\beta$ expansion (see Brihaye and Rossi 1984a). This suggests that our method is correct and relies on the higher orders we have presented.

This completes the naive MF analysis of the chiral model. The one loop correction can be computed as described in § 2, i.e. by considering the quadratic form of $S_{\rm eff}$ in the fluctuations γ and ξ (see (2.6)-(2.7)) around the saddle point

$$S^{(2)}=\sum_n\left(2\beta\sum_\mu \operatorname{Tr} E_nE_{n+\mu}-2\operatorname{Tr} E_nC_n+2\beta\sum_\mu \operatorname{Tr} D_nD_{n+\mu}-2\operatorname{Tr} B_nD_n+\mathcal{M}(B_n,C_n)\right) \tag{3.11}$$

where $\mathcal{M}$ was given in (2.17). The derivation is the same as in the large-N limit presented in Brihaye and Rossi (1984a); therefore we give only the final result:

$$\frac{F_{\rm eff}}{VN^2}(\text{one loop})=\frac{1}{2}\left(\frac{N^2-1}{N^2}(F_d(f_1)+F_d(f_1+f_2))+\frac{1}{N^2}(F_d(f_1+f_2+f_3))+F_d(f_1+g_1)\right) \tag{3.12}$$

with the notation of Guha and Lee (1983)

$$F_d(x)=\int_{-\pi}^{\pi}\left(\frac{dp}{2\pi}\right)^d\ln\left(1-2\beta x\sum_{\mu=1}^{d}\cos p_\mu\right). \tag{3.13}$$

Again, we can develop a in powers of $1/\beta d$ and obtain the following first non-trivial terms above, given respectively for U(N) and SU(N)

$$\frac{F_{\text{eff}}}{VN^2}(\text{one loop}) = \left(1+\frac{3}{N^2}\right)\left(\frac{1}{2^{11}\beta^2 d^3}+\mathrm{O}(\beta^{-3}d^{-4})\right) + \frac{1}{N^4}\left(\frac{1}{2^{12}\beta^3 d^4}+\mathrm{O}(\beta^{-4}d^{-5})\right)+\mathrm{O}\left(\frac{1}{N^6}\right) \tag{3.14}$$

$$\frac{F_{\text{eff}}}{VN^2}(\text{one loop}) = \frac{1}{2^{11}\beta^2 d^3}\left(1+\frac{1}{N^2}\left(7+\frac{1}{144}\right)\right)+\mathrm{O}(\beta^{-3}d^{-4})+\mathrm{O}\left(\frac{1}{N^4}\right) \tag{3.15}$$

to be added respectively to (3.9) and (3.10). In computing (3.14) and (3.15) we have used the fact that $f_1 = 1/2\beta d$, as can be seen from equations (2.19) and (3.3).

Next, we will use the MF approach to compute the two-point correlation functions of the chiral models. Following the spirit of MF, the integrals are dominated by the saddle point:

$$\begin{aligned}\frac{\mathrm{Tr}}{2N}\langle U_n U_m^+ + U_m U_n^+\rangle &= \frac{\mathrm{Tr}}{2N}\frac{\int \mathrm{d}V\,\mathrm{d}A(V_n V_m^+ + V_m V_n^+)\exp S_{\text{eff}}(A, V)}{\int \mathrm{d}V\,\mathrm{d}A \exp S_{\text{eff}}(A, V)} \\ &= v^2 + \text{corrections.}\end{aligned} \tag{3.16}$$

The computation of the corrections in (3.16) can only be done after diagonalising the quadratic form (2.17) in the E_n and D_n degrees of freedom. After some algebra, the following result can be obtained

$$\frac{\mathrm{Tr}}{2N}\langle V_n V_m^+ + V_m V_n^+\rangle = v^2 + \frac{2v}{\beta d}\frac{\mathrm{Tr}}{N}\langle B_j\rangle + \frac{2}{\beta}\Delta_{nm}^{-1} + \frac{\Delta_{ni}^{-1}\Delta_{mj}^{-1}}{\beta^2}\frac{\mathrm{Tr}}{N}\langle B_i B_j + C_i C_j\rangle \tag{3.17}$$

$$\Delta_{nm} = \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}p}{2\pi}\right)^d \exp(\mathrm{i}p(n-m))\sum_\mu \cos p_\mu = \int_{-\pi}^{\pi}\frac{\mathrm{d}p}{2\pi}\exp(\mathrm{i}p(n-m)\,\Delta p. \tag{3.18}$$

When restricted to the lowest order, formula (3.17) takes the form

$$\frac{\mathrm{Tr}}{2N}\langle V_n V_m^+ + V_m V_n^+\rangle = v^2 + \frac{N^2-1}{N^2}[\hat{F}(f_1)+\hat{F}(f_1+f_2)] + \frac{1}{N^2}[\hat{F}(f_1+f_2+f_3)+\hat{F}(f_1+g_1)] \tag{3.19}$$

with the definition

$$\hat{F}(x) = \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}p}{2\pi}\right)^d \frac{(\Delta_p/\Delta_0)\beta\partial x/\partial\beta + x\exp(\mathrm{i}p(n-m))}{(1-2\beta d x\Delta_p/\Delta_0)}. \tag{3.20}$$

The trivial v^2 term in (3.17) together with the first term of $\hat{F}$ provides an approximation for the asymptotic value (large separation) (Green and Samuel 1981) of the function

(3.16). The second term of $\hat{F}$ is explicitly separation dependent and decreases while the separation increases. It is easy to see that the contribution of $\hat{F}(f_1)$ is nothing other than the $1/\beta$ term obtained by perturbative methods. The interpretation of the other terms of (3.19) is less trivial and is under investigation.

4. Gauge model

The model is described by the Wilson action on a d-dimensional lattice L. The partition function is

$$Z(\beta)=\int\prod_x\prod_\mu \mathrm{d}U_{x,\mu}\exp\left(\beta N\sum_x\sum_{\mu\neq\nu}^{d}\mathrm{Tr}(U_{x,\mu}U_{x+\hat{\mu},\nu}U^+_{x+\hat{\nu},\mu}U^+_{x,\nu})\right) \tag{4.1}$$

where the matrices U belong to U(N) or SU(N), and where we pick the axial gauge $U_{x,d}=1$.

The random field transform leads to the following effective action

$$\begin{aligned}S_{\mathrm{eff}}=N\sum_x\Bigg(\beta\sum_{\mu\neq\nu}^{d-1}\mathrm{Tr}(V_{x,\mu}V_{x+\hat{\mu},\nu}V^+_{x+\hat{\nu},\mu}V^+_{x,\nu})+\beta\sum_{\mu=1}^{d-1}\mathrm{Tr}(V_{x,\mu}V^+_{x+\hat{d},\mu}+V_{x+\hat{d},\mu}V^+_{x,\mu})\\ -\sum_{\mu\neq d}^{d-1}\mathrm{Tr}(A_{x,\mu}V^+_{x,\mu}+A^+_{x,\mu}V_{x,\mu})+N\sum_{\mu\neq d}^{d-1}w(A_{x,\mu},A^+_{x,\mu})\Bigg).\end{aligned} \tag{4.2}$$

We want to perform a two-loop analysis which consists in finding the large-N behaviour of h_1, h_2, k_1 and k_2 in the following expansion of the free energy $F(\beta, d-1)$:

$$F(\beta,d-1)=(\ln Z)/VN^2 \tag{4.3}$$

$$\begin{aligned}\frac{F(\beta,d-1)}{(d-1)}=h_0(\beta,d)+\frac{1}{\beta}\left(\frac{h_1}{d-1}+\frac{k_1}{(d-1)^2}+\mathrm{O}\left(\frac{1}{(d-1)^3}\right)\right)\\ +\frac{1}{\beta^2}\left(\frac{h_2}{(d-1)^2}+\frac{k_2}{(d-1)^3}+\mathrm{O}\left(\frac{1}{(d-1)^4}\right)\right)+\mathrm{O}(\beta^{-3}).\end{aligned} \tag{4.4}$$

Things are arranged so that $\beta(d-1)$ is the effective parameter of the expansion. The naive MF analysis will provide the h_1 and h_2 coefficients, while k_1 and k_2 receive a contribution from naive and improved MF; this is due to the fact that there is a gauge fixing term in the action. Indeed, in the case of the chiral model, only the variable βd appears at the naive MF level (see (3.9)-(3.10)).

4.1. Naive mean field

With the ansatz (2.5) for the naive MF approach, the effective action (4.2) becomes

$$S^{\mathrm{naive}}_{\mathrm{eff}}=N^2V(d-1)\{\beta[(d-2)v^4+2v^2]-2av+w(a)\} \tag{4.5}$$

and the saddle-point equations are

$$2\beta(d-2)v^3+2\beta v=a \tag{4.6}$$

$$v=\tfrac{1}{2}\dot{w}(a)=1-\frac{1}{4a}+\frac{1}{2N^2}\left(-\frac{1}{8a(2a-1)}+\delta\dot{s}(a)\right)+\mathrm{O}\left(\frac{1}{N^4}\right) \tag{4.7}$$

where the dot denotes differentiation with respect to a, and $s(a)$ is given in equation (2.27).

To solve the equations (4.6)-(4.7), we assume, as for the chiral model, that a and v admit an $1/N^2$ expansion of the form (3.5). Clearly, a_0 and v_0 obey the set of equations appearing in Hasegawa and Yang (1983):

$$2\beta v_0[(d-2)v_0^2+1]=a_0$$

$$v_0=1-1/4a_0 \tag{4.8}$$

which cannot be solved exactly. However, the coefficients a_n and v_n can be computed in terms of a_0 and v_0; for $n=1$, one has

$$v_1=\frac{1}{2}\left(\frac{4a_0^2}{4a_0^2-2\beta[3(d-2)v_0^2+1]}\right)\left(\frac{-1}{8a_0(2a_0-1)}+\delta\dot{s}(a_0)\right) \tag{4.9}$$

$$a_1=2\beta v_1[3(d-2)v_0^2+1]. \tag{4.10}$$

Now, the effective action (4.5) can be $1/N^2$ expanded, and the result becomes very simple after the use of the saddle-point equations (4.6):

$$\frac{S_{\text{eff}}^{\text{naive}}}{N^2V(d-1)}=\beta v_0^2-\tfrac{3}{2}(a_0-\tfrac{1}{4})+w_\infty(a_0)+\frac{1}{N^2}w_1(a_0)$$
$$+\frac{1}{N^4}(w_2(a_0)+\tfrac{1}{2}a_1\dot{w}_1(a_0))+\mathrm{O}\left(\frac{1}{N^6}\right). \tag{4.11}$$

All that we have done up to now is quite formal as far as we do not know a_0. Let us then give an approximation of a_0 by looking at the equations (4.8) which determine v_0 as a function of β in three different branches. The condition $a_0\geqslant\frac{1}{2}$ (or $v_0\geqslant\frac{1}{2}$) selects the upper branch, where we have the following approximations ($\tilde{\beta}\equiv 8(d-1)\beta$):

$$v_0=1-\frac{1}{\tilde{\beta}}-\frac{3d-5}{\tilde{\beta}^2(d-1)}-\frac{15d^2-51d+44}{\tilde{\beta}^3(d-1)^2}+\mathrm{O}(\tilde{\beta}^{-4}) \tag{4.12}$$

$$a_0=\tfrac{1}{4}\tilde{\beta}-\frac{3d-5}{4(d-1)}+\frac{1}{4\tilde{\beta}(d-1)^2}(-6d^2+21d-19)$$
$$-\frac{1}{4\tilde{\beta}^2(d-1)^3}(28d^3-148d^2-34d-162)+\mathrm{O}(\tilde{\beta}^{-3}) \tag{4.13}$$

on the domain $\beta_c<\beta\leqslant\infty$, where β_c is the point where the tangent to v becomes vertical:

$$\partial\beta/\partial v_0|_{v_0\text{crit}}=0\Rightarrow(d-2)(-4v_{0c}^3+3v_{0c}^2)-2v_{0c}+1=0. \tag{4.14}$$

We present some numerical results about β_c.

Table 1.

d	v_{0c}	β_c
2	0.5	0.5
3	0.62	0.38
4	0.659	0.29
large	$\frac{3}{4}-\frac{2}{9(d-2)}+\dots$	$\frac{32}{27(d-2)}+\dots$

There is an important point to be stressed here: it is a fact that for $d \geq 3$ and on the inverval $[\beta_c, \infty[$, the functions a_0 and v_0 never approach the value $\frac{1}{2}$; as a consequence, the $1/N^2$ corrections in (4.11) are defined on the whole domain $[\beta_c, \infty[$. The situation is different in the chiral model where the $1/N^2$ corrections become infinite whenever β approaches the critical value β_c in such a way that no valuable information can be obtained in the neighbourhood of β_c.

Now, we can exploit our knowledge of all quantities entering in formula (4.11) and give the expansion in $1/\beta$ that we have found. We will stop the expansion to $1/\beta^2$ terms. A quick analysis shows that to this order of perturbation, we only need the $1/N^2$ approximation in (4.11). Straightforward algebra leads to the following expression for S_{eff} ($A = 0.248\,75$)

$$\frac{S_{\text{eff}}^{\text{naive}}}{N^2V(d-1)} = \beta d - \frac{1}{2}\left(1-\frac{\delta}{N^2}\right)\ln\frac{\tilde{\beta}}{2} - \frac{3}{4} + \frac{1}{N^2}\left(\delta \ln 2\pi^2 + \tfrac{1}{12} - A - \tfrac{1}{12}\ln N\right)$$
$$+ \frac{1}{2^2\tilde{\beta}}\left[3 - \frac{2}{d-1} + \frac{1}{N^2}\left(1 - 4\delta\left(2 - \frac{1}{d-1}\right)\right)\right] \tag{4.15}$$
$$+ \frac{1}{\tilde{\beta}^2}\left[\frac{7}{4} - \frac{5}{2(d-1)} + \frac{1}{2N^2}\left(2 - \frac{1}{d-1}\right) + \frac{\delta}{N^2}\left(-\frac{43}{6} + \frac{17}{2(d-1)}\right)\right]$$
$$+ O(\tilde{\beta}^{-3}) + O\left(\frac{1}{N^4}\right).$$

4.2. *Improved mean field*

The aim is to substitute for $V_{x,\mu}$ and $A_{x,\mu}$ the expressions (2.6)-(2.7) in the effective action (4.2), and to compute the determinant of the quadratic form in the variables ξ and γ. This tedious task has been accomplished by Müller and Rühl (1982) and Hasegawa and Yang (1983). However, we can develop further their formal results by taking into account the $1/N^2$ expansion of w, a and v given in the previous section, and put them in a form suitable to complete (4.15). The contribution of the degree of freedom of type B can be easily computed using the formula (53) of Müller and Rühl (1982). We find, up to the order we are interested in:

$$\frac{S_{\text{imp}}^{\text{I}}}{N^2V(d-1)} = \frac{3}{2^5\tilde{\beta}^2(d-1)}\left(1 + \frac{3}{N^2} + \frac{4\delta}{N^2}\right) + O(\tilde{\beta}^{-3}) + O\left(\frac{1}{N^4}\right) \tag{4.16}$$

where the terms in the brackets come from the contribution of the traceless part and the trace of B.

Now, the treatment of the C type degree of freedom requires more care. A tedious computation gives, up to the order of interest

$$\frac{S_{\text{imp}}^{\text{II}}}{N^2V(d-1)} = \frac{1}{(d-1)}\left(1-\frac{\delta}{N^2}\right)\left[\tfrac{1}{2}[\pi \ln 2 - \tfrac{3}{4} + \ln(d-1)] - \frac{1}{\tilde{\beta}}\left(1-\frac{\delta}{N^2}\right)\right.$$
$$\left. + \frac{1}{2\tilde{\beta}^2}\left(7 - \frac{16\delta}{N^2} + \frac{1}{N^2}\right)\right] + O(\tilde{\beta}^{-3}) + O\left(\frac{1}{N^4}\right). \tag{4.17}$$

This last result is based on the fact that for $U(N)$, the eigenvalues linked to the traceless part and the trace of C are equal, and moreover, the leading contribution of the eigenvalue related to the trace of C is zero in the $SU(N)$ case.

It is worth noticing that the corrections (4.16) are small and seem to converge rapidly in $[\beta(d-1)]^{-1}$. On the contrary, their analogue in equation (4.17) are more important, including a term independent of β. This is due to the peculiar $\beta(d-1)$ dependence of the eigenvalue f_1 linked to these degrees of freedom. The situation is much simpler in the chiral model where the corresponding contribution reduces to a constant (see the first term in (3.12)). In the formula (4.17), only the terms with relevant order in $\beta(d-1)$ were kept.

Collecting the results (4.15), (4.16) and (4.17), we have, using (4.4)

$$h_1=\frac{1}{2^5}\left(3+\frac{1}{N^2}(1-8\delta)\right)+O\left(\frac{1}{N^4}\right) \tag{4.18a}$$

$$h_2=\frac{1}{2^6}\left[\frac{7}{4}+\frac{1}{N^2}\left(1-\frac{43\delta}{6}\right)\right]+O\left(\frac{1}{N^4}\right) \tag{4.18b}$$

$$k_1=\frac{1}{2^4}\left(-3+\frac{4\delta}{N^2}\right)+O\left(\frac{1}{N^4}\right) \tag{4.18c}$$

$$k_2=\frac{1}{2^{11}}\left(35+\frac{9}{N^2}-\frac{84\delta}{N^2}\right)+O\left(\frac{1}{N^4}\right) \tag{4.18d}$$

$$h_0=\beta d-\frac{1}{2}\left(1-\frac{\delta}{N^2}\right)\left(\ln 4\beta(d-1)-\frac{1}{d-1}(\pi\ln 2-\tfrac{3}{4}+\ln(d-1))\right)$$
$$-\frac{3}{4}+\frac{1}{N^2}\left(\frac{\delta}{2}\ln 2\pi^2+\tfrac{1}{12}-A-\tfrac{1}{12}\ln N\right). \tag{4.18e}$$

We then have an expression for the free energy of the Wilson gauge model, computed using the MF technique. It gives as announced the $1/\beta$ and $1/\beta^2$ coefficients up to $1/N^4$ corrections in $U(N)$ and $SU(N)$.

5. Conclusions

In this paper, we have tried to incorporate the $1/N^2$ expansion in the mean-field approach of lattice chiral and gauge models. We have verified to the best of our knowledge that the double $1/d$, $1/N^2$ expansion is in agreement with the standard weak-coupling expansion of the free energy; this relies on the higher orders we have computed (β^{-2} for gauge models, β^{-3} for chiral models) and, more generally, on the method we have used.

One may hope that, with the increased precision, our approximation for the free energy can be considered over a large domain of the coupling constant, i.e. a domain covering part of the (delicate) intermediate coupling region.

In this spirit, we have argued that for the gauge model the $1/N^2$ expansion is valid in the intermediate (or critical point) region. This is unlike the chiral model, where the $1/N^2$ correction becomes infinite at the critical point, providing a limitation to the procedure developed above.

Beside any gain of accuracy, our results allows for a comparison between the (β depending) coefficients in the double expansion we have presented. In particular, each of them is (up to $\ln\beta$ term) an analytic function of β^{-1}.

With this motivation we have attempted to understand the weak-coupling behaviour of chiral models by exploiting MF to compute the two points correlation function. The naive MF rely on the asymptotic value (large separation) only while the asymptotic decay is related, in leading order in d, to the improved MF.

Finally, we have presented the cubic fluctuations of the single-link integral which contribute to the tadpole diagram of $\langle B \rangle$ in equation (3.16).

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Appendix

In this appendix, we present the definitions of the functions f_i coming in the cubic fluctuations of the one-link single integral:

$$\frac{f_5}{N}\delta_{kl}\delta_{lm}+\frac{f_6}{N^2}(\delta_{kl}+\delta_{lm}+\delta_{km})+\frac{f_7}{N^3}=4a^3\frac{\partial^3}{\partial\lambda_k\,\partial\lambda_l\,\partial\lambda_m}w\bigg|_{\text{sp}} \tag{A1}$$

$$\frac{f_8}{N}(\delta_{km}+\delta_{lm})+\frac{f_9}{N^2}=4a^3\frac{\partial}{\partial\lambda_m}\left(\frac{\partial_{\lambda_k}-\partial_{\lambda_l}}{\lambda_k-\lambda_l}\right)w\bigg|_{\text{sp}} \tag{A2}$$

$$\frac{f_{10}}{N}=4a^3\left(\frac{1}{(\lambda_l-\lambda_k)(\lambda_m-\lambda_k)}\frac{\partial}{\partial\lambda_k}+(\text{permutations})\right)w\bigg|_{\text{sp}} \tag{A3}$$

$$g_2=N\left\{\frac{1}{2a^3}\frac{\partial^2}{\partial\phi^2}+\frac{1}{a}\frac{\partial^3}{\partial\phi^2\,\partial\lambda_i}\right\}w\bigg|_{\text{sp}}. \tag{A4}$$

It is worth noticing that

$$f_5(a)=2f_8(a)=2f_{10}(a)=\frac{3}{a^2}-\frac{5}{2^2a^3}-\frac{15}{2^4N^2a^3(2a-1)}+\frac{15\delta}{2N^2a^2}\dot{s}(a) \tag{A5}$$

$$f_6(a)=f_9(a)=-\frac{1}{2^2a^3}-\frac{3}{2^4N^2a^3(2a-1)}+\frac{3\delta}{2^3N^2a^4}\left(\ddot{s}(a)+\frac{2}{a}\dot{s}(a)\right) \tag{A6}$$

$$f_7(a)=-\frac{1}{2^4N^2a^3(2a-1)^3}+\frac{\delta}{2^2N^2a^2}\left(\ddot{s}(a)+\frac{2}{a}\dot{s}(a)\right). \tag{A7}$$

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